1092 Calculus4 ME Final Exam

June 19, 2021

There are FIVE questions in this examination.

1. Let $V = \text{span}\left\{\begin{bmatrix}3\\-4\\-5\\-1\end{bmatrix}, \begin{bmatrix}4\\-2\\-2\\1\end{bmatrix}, \begin{bmatrix}1\\3\\2\\3\end{bmatrix}, \begin{bmatrix}-2\\2\\-4\\1\end{bmatrix}\right\}$ and $W = \text{span}\left\{\begin{bmatrix}1\\1\\-1\\-1\end{bmatrix}, \begin{bmatrix}1\\-2\\-2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\2\\0\end{bmatrix}\right\}$.

   (a) (5 pts) Find the dimension of the vector subspace V.

   (b) (5 pts) Find a basis for W. (Hint: a basis is a linearly independent set of vectors that span W)

   (c) (2 pts) Show that V and W are not equal.

2. Let $A = \begin{bmatrix}-1 & 3 & 1\\3 & -1 & 1\\1 & 1 & 1\end{bmatrix}$.

   (a) (8 pts) Find the eigenvalues of A and corresponding eigenvectors given that $\det(A - \lambda I_3) = -\lambda^3 - \lambda^2 + 12\lambda$.

   (b) (4 pts) Diagonalize A. That is, find an orthogonal matrix P (i.e. $P^T P = I_3$) and a diagonal matrix D such that $P^T AP = D$.

   (c) (4 pts) Determine whether $A + 5I_3$ is positive definite, negative definite, or indefinite.

   (d) (4 pts) Determine whether $A - 5I_3$ is positive definite, negative definite, or indefinite.

3. Maximize $f(x, y, z) = yz$ subject to $x + z = 1$, $x^2 + y^2 \leq 6$, $z \geq 0$.

   (a) (4 pts) Check whether the NDCQ is satisfied.

   (b) (8 pts) Write out the Lagrangian function and the first order conditions.

   (c) (8 pts) Solve the constrained optimization problem given that the constraints form a closed and bounded region.

4. Suppose that you keep t hours a day as leisure time and $16 - t$ hours to tutor with wage 400 dollars per hour. Your daily budget is $200+400(16-t)$ and you spend money on food and clothes with prices 250 and 350, respectively, per unit. If you consume x units of food and y units of clothes, then your utility function $U(x, y, t)$ depends on x, y and hours of leisure time t, where $\frac{\partial U}{\partial x} > 0$, $\frac{\partial U}{\partial y} > 0$, and $\frac{\partial U}{\partial t} > 0$. Now you want to maximize $U(x, y, t)$ under the constraints $250x + 350y \leq 200 + 400(16 - t)$, $t \leq 16$, $t \geq 0$, $x \geq 0$, $y \geq 0$.

   (a) (8 pts) Write down the Kuhn-Tucker Lagrangian function, $\tilde{L}(x, y, t, \lambda_1, \lambda_2)$, and the first order conditions in the Kuhn-Tucker formulation.

   (b) (4 pts) Show that if (x^*, y^*, t^*) is a maximizer, then the constraint $250x+350y \leq 200+400(16-t)$ is binding at (x^*, y^*, t^*).

   (c) (6 pts) Show that if (x^*, y^*, t^*) is a maximizer satisfying $x^* > 0$, $y^* > 0$, and $0 < t^* < 16$, then

$$\frac{\partial U}{\partial t}(x^*, y^*, t^*)\frac{1}{400} = \frac{\partial U}{\partial x}(x^*, y^*, t^*)\frac{1}{250} = \frac{\partial U}{\partial y}(x^*, y^*, t^*)\frac{1}{350}.$$

5. Consider the problem of maximizing $f(x, y, z) = xyz$ subject to $2x + y + z = 18$, and $x + 2y + z = 18$.

   (a) (1 pts) Write down the Lagrangian function for this problem, $L(x, y, z, \mu_1, \mu_2)$, where μ_1 and μ_2 are the Lagrange multipliers.

   (b) (2 pts) Check whether the NDCQ is satisfied.

   (c) (4 pts) Write down the first order conditions for this problem.

   (d) (7 pts) Show that the solution of the first order conditions are $(x, y, z, \mu_1, \mu_2) = (4, 4, 6, 8, 8)$ or $(x, y, z, \mu_1, \mu_2) = (0, 0, 18, 0, 0)$. (You have to show your steps to get complete credits).

   (e) (7 pts) Check the second order conditions at $(x, y, z, \mu_1, \mu_2) = (4, 4, 6, 8, 8)$ and $(x, y, z, \mu_1, \mu_2) = (0, 0, 18, 0, 0)$. Show that $(4, 4, 6)$ a local maximizer and $(0, 0, 18)$ is a local minimizer.

   (f) (1 pt) Does $f(x, y, z) = xyz$ have a global maximum or global minimum subject to $2x+y+z = 18$, and $x + 2y + z = 18$?

   (g) (4 pts) Estimate the value of the local maximum of the following function $f(x, y, z) = xyz$ subject to $2x + y + z = 18.1$, and $x + 2y + z = 18.2$.

   (h) (4 pts) Estimate the value of the local maximum of the following function $f(x, y, z) = xyz+0.1x$ subject to $2x + y + 1.1z = 18$, and $x + 2y + z = 18.1$.